A nonlinear model having solutions akin to Abrikosov's mixed state

2002 J. Phys. A: Math. Gen. 35 2005

(http://iopscience.iop.org/0305-4470/35/8/314)

J. Phys. A: Math. Gen. **35** (2002) 2005–2011 PII: S0305-4470(02)29263-X

A nonlinear model having solutions akin to Abrikosov's mixed state

G P Malik[1] **and Usha Malik**[2]

[1] Theory Group, School of Environmental Sciences, Jawaharlal Nehru University, New Delhi-110067, India
[2] Physics Department, Miranda House, University of Delhi, Delhi-110007, India

E-mail: gp7malik@bol.net.in

Received 21 September 2001, in final form 11 December 2001
Published 15 February 2002
Online at stacks.iop.org/JPhysA/35/2005

Abstract
We present new exact solutions of a model that was originally proposed in the context of solitons and show that they possess features akin to Abrikosov's solution of the Ginzburg–Landau equations.

PACS numbers: 74.20.-z, 02.30.-f, 05.45.Yv, 74.60.Ec

We present here, in the static limit, new exact solutions of a model [1] defined by equations

$$\begin{aligned} \sigma'' &= -\sigma + \sigma^3 + d\rho^2\sigma \\ \rho'' &= f\rho + \lambda\rho^3 + d\rho(\sigma^2 - 1) \end{aligned} \tag{1}$$

where a prime denotes a derivative with respect to x, σ and ρ are real scalar fields and f, λ and d are parameters. Further, we draw attention to the possible use of these solutions in the domain of superconductivity: this is done by showing that for appropriately chosen values of the parameters of the model, at least one of these solutions is akin to Abrikosov's classic solution [2] of the Ginzburg–Landau equations [3].

The backdrop of the model is as follows. The model was proposed [1] at a time when it seemed rather attractive to think of an elementary particle as a soliton, leading one to hope that solitons would pervade much of physics. This naturally led to a study of coupled field theories that admitted more than one soliton solution and enabled one, among other things, to study the nature of forces operative between solitons. More specifically, from the same vantage point, a study of the structure of soliton theories with some internal symmetry was a step forward, and it is this consideration that led to the proposal of the above model. Similar investigations, carried out around the same time that the model was proposed, may also be found in [4, 5] where it is pointed out that the existence of a conserved charge in such theories can provide a non-topological stability mechanism for the soliton solutions. The search for soliton solutions in the physical world also led one to investigate similarly coupled theories in three space dimensions [5].

0305-4470/02/082005+07$30.00

The new solutions of set (1) are in terms of Jacobian elliptic functions (JEFs). They hold without any constraints on the parameters λ, f and d, and are therefore the most general solutions of the set. Since an elliptic function is a doubly periodic meromorphic function, the translation of its primitive cell in the Argand diagram generates a lattice that is reminiscent of the lattice associated with Abrikosov's solution [2] of the Ginzburg–Landau equations [3] of superconductivity. The correspondence between our solutions and Abrikosov's mixed state is sharpened below.

Let us begin by noting that, at the time of their proposal, no exact solutions of (1) were given. A solution, without derivation, was given in [6]:

$$\begin{aligned}\sigma(x) &= \tanh(\sqrt{f}x)\\ \rho(x) &= (1-2f)^{1/2}\operatorname{sech}(\sqrt{f}x)\end{aligned}\tag{2}$$

which is valid for $\lambda = d = 1$. This led to a study aimed at finding solutions of set (1), and sets similar to it, in a systematic manner and resulted in a method that was called the method of trial orbits [7]. Thus, three different sets of solutions for the above equations were found, valid for different constraints among the parameters of the model. These solutions include topological as well as non-topological solitons, and yield the solutions given by (2) as a special case. Subsequently, based on a virial theorem for solitons [5], another method of finding solutions of (1) was given in [8], leading to an enlargement of the set of solutions already known.

We turn now to derivation of the new solutions. Taking into account the structure of (1), and noting that [9]

$$(\mathrm{d}^2/\mathrm{d}u^2)sn^n(u,k) = n(n-1)sn^{n-2}(u,k) - n^2(1+k^2)sn^n(u,k) + n(n+1)k^2sn^{n+2}(u,k)\tag{3}$$

(11 similar relations hold for the remaining JEFs), we are led to the following ansatz as a solution-set:

$$\sigma(x) = Asn(\alpha x,k) \qquad \rho(x) = Bcn(\alpha x,k).\tag{4}$$

Equations (4) and (1) imply

$$\begin{aligned}f - d + A^2d &= \alpha^2(2k^2-1)\\ \lambda B^2 - A^2d &= -2\alpha^2k^2\\ A^2 - B^2d &= 2\alpha^2k^2\\ B^2d - 1 &= -\alpha^2(k^2+1)\end{aligned}\tag{5}$$

which are easily solved to give

$$\begin{aligned}A^2 &= 2(\lambda-d)(f-d+1)/D\\ B^2 &= 2(f-d+1)(d-1)/D\\ \alpha^2 &= N/D\\ k^2 &= [2(d-f-1)+\lambda(f-d+1)]/N\end{aligned}\tag{6}$$

where

$$D = d(d-2) + \lambda(3-2d) \qquad \text{and}$$
$$N = d(d-2)(d-f) - \lambda(d+f-2).$$

Before we discuss this solution further, let us note that the 12 JEFs can be divided into four families of co-polar functions [10]; thus, *sn, cn*, and *dn* belong to one family, *cd, sd*, and *nd* to another family, and so on. The general nature of relation (3) then suggests that any two different functions from the same co-polar family should solve equations (1), which is indeed found to be so. Thus,

$$\sigma(x) = Csn(\beta x,k') \qquad \rho(x) = Ddn(\beta x,k')\tag{7}$$

are also solutions of (1), with C, D, β, and k' given by expressions similar to those in (6). Let us quote here a result from the *cd, sd*, and *nd* family; for

$$\sigma(x) = A_1 cd(\alpha_1 x, k_1) \qquad \rho(x) = B_1 sd(\alpha_1 x, k_1) \tag{8}$$

we have

$$\begin{aligned} A_1^2 &= 2(-f\lambda - \lambda - d^2 + d\lambda + d + fd)/D_1 \\ B_1^2 &= 2(f\lambda - 2f^2\lambda d + 2f^2\lambda - 3fd\lambda + d^2 - 2fd - \lambda - 2f^2 d \\ &\qquad + 2f^2d^2 - 3fd^3 + 2\lambda f d^2 + d^4 - 2d^3 + 2d\lambda - \lambda d^2 + 5fd^2)/N_1 D_1 \\ \alpha_1^2 &= N_1/D_1 \\ k_1^2 &= (fd^2 - f\lambda - \lambda + d^2 + d\lambda - d^3)/D_1 \end{aligned} \tag{9}$$

where

$$D_1 = (-3\lambda + 2d - d^2 + 2d\lambda) \qquad \text{and}$$
$$N_1 = (f\lambda - 2\lambda + 2d^2 - 2fd + d\lambda - d^3 + fd^2).$$

The periodicities of the above solutions are calculable from the values of the parameters k, k', or k_1 which, without loss of generality, can be assumed to be real and to lie between zero and unity because of relations such as [11]

$$\begin{aligned} &sn(u, \mathrm{i}k) = [1/(1+k^2)^{1/2}] sd[u(1+k^2)^{1/2}, k/(1+k^2)^{1/2}] \qquad \text{and} \\ &sn(u, k) = (1/k) sn(ku, 1/k). \end{aligned} \tag{10}$$

It follows from (10) that the solution-sets of (1) in terms of JEFs are not independent: a *sn–cn* pair solution can be converted into a *sn–dn* pair solution when $k > 1$, to a *sd–cd* pair solution when k is pure imaginary (and so on), with a definite relationship between the arguments and the parameters of the two pairs. It also follows from (10) that the periods of any solution-set are real [10].

We note that the solution corresponding to type-B orbit in [7] is obtainable from our approach by simply choosing $A = 1$ in the ansatz of (4). The set (5) now has only three unknowns; it follows that the parameters λ, d and f cannot all be independent. Solving the set for B, α, k and λ, we now obtain

$$\begin{aligned} B^2 &= (1 - 2f)/d \\ \alpha^2 &= f \\ k^2 &= 1 \\ \lambda &= d(2f - d)/(2f - 1) \end{aligned} \tag{11}$$

and since $sn = \tanh$, and $cn = \mathrm{sech}$ when $k^2 = 1$, our ansatz reduces to the solutions given in equations (12) of [7]; further specialization to the case $\lambda = d = 1$ then yields the solutions given in (2), as already noted. There are two further properties of the solution-set (4) that are particularly interesting: (a) as k^2 tends to zero, the functions *sn* and *cn* tend to sine and cosine respectively, and (b) all through their domain of definition, these functions remain 'out of phase', i.e. *sn* (or tanh or sine) has its maximum value when *cn* (or sech or cosine) has its minimum value.

Let us now draw attention to the most important features of a type-II superconductor in a magnetic field, features that were unravelled through an approximate solution of Ginzburg–Landau equations [3] obtained by Abrikosov [2] in his monumental work. These are: (a) there are two critical magnetic fields, Hc_1 and Hc_2 ($Hc_1 < Hc_2$) for such a superconductor, apart from the usual thermodynamic critical field. (b) When the magnetic field, H, exceeds Hc_2, the superconductor behaves as a normal metal. (c) When H is less than Hc_1, the superconductor behaves as a type-I superconductor (exhibits perfect Meissner effect;

equivalently, the Ginzburg–Landau order parameter becomes unity). (d) For $Hc_1 < H < Hc_2$, the superconductor assumes a state that is a fine-scale mixture (mixed state) of superconducting and normal regions. The overall arrangement of these regions has the structure of a doubly periodic lattice in which the vortices of superconducting electrons enclose normal regions. (e) The vortices become sparser as one proceeds from Hc_2 to Hc_1, and (f) the order parameter and the magnetic field are out of phase, in the sense that magnetic field is least when the order parameter has its maximum value.

Considerations of the two preceding paragraphs suggest that the solution-set of (σ, ρ)-model given in (4) might embody the essence of the mixed state. For further exploration, we need the explicit form of Ginzburg–Landau equations [3] written in terms of a function $F(x)$ and the vector potential $\boldsymbol{A}(x, y)$ defined as

$$\psi(x, y) = \mathrm{e}^{\mathrm{i}ky} F(x)$$
$$\boldsymbol{A}(x, y) = A(x, y)\boldsymbol{j}.$$

These lead to

$$\begin{aligned} &(1/\chi 2)F'' = u^2 F - F + F^3 \\ &u'' = uF^2 \end{aligned} \tag{12}$$

where

$$u = (A - k/\chi).$$

These equations are exact. The well-known classic solution of an approximate form of these equations is in terms of a theta-function and is given by [2]

$$\psi(x, y) = \text{const} \times \exp(-\chi^2 x^2)\theta_3(v, \tau) \tag{13}$$

where

$$v = \mathrm{i}\chi(x + \mathrm{i}y)/\surd(2\pi)$$
$$\tau = \mathrm{i}$$

and the notation used is that of [11]. In figure 1, we have given a plot of $|\psi|^2$ for $\chi = 7.5$ for $-0.34 \leqslant x, y \leqslant 0.34$, the step size being approximately 0.02. The value of χ determines the periodicities of the function; changing it does not change the basic features of the plot. The value chosen here is for numerical convenience later. Figure 1 is, of course, similar to the one given in Abrikosov's original paper [2] (where χ is not specified).

We now observe that set (1) goes over to set (12) if

$$\begin{aligned} &d = 1/\chi^2 \qquad f = d \qquad \lambda = 0 \qquad \sigma = F \\ &\rho = \chi u \qquad x = \chi x \qquad \mathrm{d}/\mathrm{d}x = (1/\chi)\,\mathrm{d}/\mathrm{d}x. \end{aligned} \tag{14}$$

Indeed, these replacements cause N to become equal to zero, whence k^2 becomes infinite (see (6)). We now choose to explore the nature of solutions of set (1) for λ not exactly equal to zero, but close to it.

For values of χ and λ close to 7.5 and *zero* respectively, one may be led to six different cases: α either real or imaginary, for each of which $m(= k^2)$ can lie between zero and one, be greater than one, or be negative. We report here the findings for one of these cases, namely imaginary α and $0 \leqslant m \leqslant 1$. Choosing $\chi = 7.5$ and $\lambda = 0.003$, (6) and (14) yield

$$A = 1.059\,158 \qquad \alpha = 0.472\,957\mathrm{i} \qquad m = 0.455\,421 \qquad d = f = 0.017\,778.$$

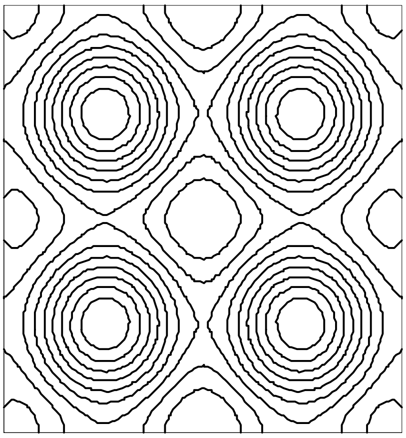

Figure 1. Abrikosov's contour plot of $|\psi(x, y)|^2$ for $\chi = 7.5$, vide equation (13).

Knowing m and $m_1 = 1 - m$, we can calculate numbers K and K_1, and nomes q and q_1 defined below:

$$K = \int_0^{\pi/2} \mathrm{d}\theta/(1 - m\sin^2\theta)^{1/2} = 1.818\,044$$

$$K_1 = \int_0^{\pi/2} \mathrm{d}\theta/(1 - m_1\sin^2\theta)^{1/2} = 1.893\,805$$

$$q = \exp(-\pi K_1/K) = 0.037\,911$$

$$q_1 = \exp(-\pi K/K_1) = 0.049\,001.$$

We now let $x \to z = x + \mathrm{i}y$ in (4) and, using an appropriate formula [10] for elliptic function of a complex argument, separate out the real and imaginary parts of $\sigma(z)$:

$$\sigma(z) = A(s_1 d_2 + \mathrm{i}c_1 d_1 s_2 c_2)/(c_2^2 + m s_1^2 s_2^2) \tag{15}$$

where

$$s_1 = sn(-\beta\chi y, m)$$

$$s_2 = sn(\beta\chi x, m_1), (\beta = |\alpha|)$$

and c_1, c_2 and d_1, d_2 denote cn and dn functions respectively, with the same arguments and parameters as for s_1 and s_2. One may now evaluate the right-hand side of (15) by employing expansions of the elliptic functions in terms of trigonometric functions [10]; alternately, one may use relations between elliptic functions and theta functions and employ the expansions for the latter functions (which also involve trigonometric functions, but are more rapidly convergent). For computational convenience we have chosen here to follow the second option after verifying that the first option gives the same results.

It is straightforward to show that, in terms of theta functions [10], (15) yields

$$\sigma(z) = A(m_1/m)^{1/2}[\mathrm{Num}/\mathrm{Den}] \tag{16}$$

where

$$\mathrm{Num} = \theta_1(s)\theta_3(u)\theta_4(s)\theta_4(u) + \mathrm{i}\theta_1(u)\theta_2(s)\theta_2(u)\theta_2(s)$$

$$\mathrm{Den} = [\theta_2(u)\theta_4(s)]^2 + [\theta_1(s)\theta_1(u)]^2$$

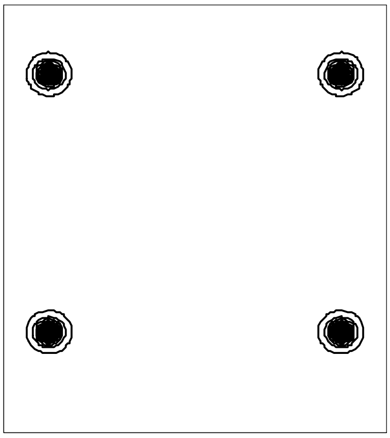

Figure 2. Contour plot of $|\sigma(z)|^2$ for $\chi = 7.5$ and $\lambda = 0.003$, vide equation (16). For details of contours between the vortices, see figure 3.

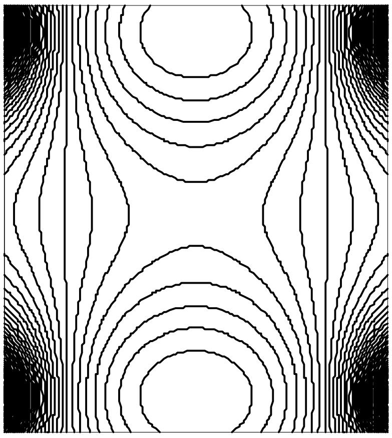

Figure 3. Contour plot of $|\sigma(z)|^2$ for $\chi = 7.5$ and $\lambda = 0.003$ with details of contours between the vortices.

where

$$\begin{aligned}
\theta_1(s) &= 2\sum(-1)^{n+1}q^{(n-0.5)^\wedge 2}\sin((2n-1)s)\\
\theta_2(s) &= 2\sum q^{(n-0.5)^\wedge 2}\cos((2n-1)s)\\
\theta_3(s) &= 1+2\sum q^{n^\wedge 2}\cos(2ns)\\
\theta_4(s) &= 1+2\sum(-1)^n q^{n^\wedge 2}\cos(2ns)(1\leqslant n\leqslant\infty)
\end{aligned}$$

where

$$s = -(\beta y\chi)[\pi/(2K)]$$

and theta-functions of argument u are given by similar expansions except that q is replaced by q_1 and u is given by

$$u = (\beta x\chi)[\pi/(2K_1)].$$

Using (16), we give plots of $|\sigma(z)|^2$ in figures 2 and 3 for the numerical values given above. Figure 2 is qualitatively similar to the plot obtained by [2] for the same value of χ (see figure 1), except that vortices of finite height in the latter have been replaced by vortices of infinite height in the former. This feature causes the details of contours between the vortices to be lost, as was to be expected because Abrikosov's solution is in terms of a θ-function (which is analytic throughout the finite part of the z-plane [9]), whereas our solution is in terms of JEFs (each of which is analytic except at the location of its simple poles, for example, points congruent to $\mathrm{i}K_1$ or to $2K+\mathrm{i}K_1$, for the *sn*-function) [9]). We note that figure 2 has been drawn for $-0.7 \leqslant x \leqslant 0.7$ and $-0.4 \leqslant y \leqslant 1.3$; the step size being approximately 0.02. In order to bring out the detailed structure of contours between the vortices, one might plot the same function by suitably restricting the variables x and y. Figure 3 is such a plot, where $-0.4 \leqslant x \leqslant 0.4$ and $-0.1 \leqslant y \leqslant 1.1$, the step size being 0.013.

We conclude with the following remarks:

(1) It seems appropriate to begin with a remark about solitons. Let set (1) be the description of a system at a given time with definite values of the parameters 'λ', 'd' and 'f'. Then, we have shown here that the general solution of the set is in terms of JEFs the parameters

($m = k^2$) of which depend on the values of 'λ', 'd' and 'f'. Further, if the system is embedded in an environment subject to thermal/magnetic changes, then 'λ', 'd' and 'f' will change, leading to changes in the values of the parameters m. Thus, a soliton solution is seen to be a descendant of a doubly periodic solution that has lost one of its periodicities (this happens when $m = 1$).

(2) We have reported here the results of the model for one of the six sets of values that α and m can have namely, imaginary α and $0 \leqslant m \leqslant 1$, pertaining to $\chi = 7.5$ and $\lambda = 0.003$. For the same value of χ, if λ is progressively reduced, m becomes negative around $\lambda = 0.0001$, while α remains imaginary. The nature of plots in this case is similar to those for the case reported here, except that periodicities of the plotted function become larger (this makes numerical exploration a bit harder). On the other hand, if λ is progressively increased (for the same value of χ), α becomes real around $\lambda = 0.012$, while m remains between 0 and 1. This case seems to lead to doubly periodic clusters of spikes, rather than single spikes. Each cluster here consists of a tall spike surrounded by lower spikes.

(3) Since the model studied seems to be capable of yielding a variety of doubly periodic structures, it may be well worth a detailed study in its own right—beyond the limit in which it reduces to the Ginzburg–Landau equations. We propose to undertake such a study soon.

Acknowledgments

GPM would like to acknowledge having had enjoyable conversations concerning Abrikosov's solution of Ginzburg–Landau equations with Professor L K Pande. He would also like to thank Professor R Rajaraman for clarification about one of the solutions given in his paper [7], and Professor V S Varma for a valuable discussion.

References

[1] Rajaraman R and Weinberg E J 1975 *Phys. Rev.* D **11** 2950
[2] Abrikosov A A 1957 *Sov. Phys.–JETP* **5** 1174
[3] Saint-James D, Sarma G and Thomas E J 1969 *Type II Superconductivity* (Oxford: Pergamon)
[4] Lee T D 1976 *Phys. Rep.* **23** 254
[5] Friedberg R, Lee T D and Sirlin A 1976 *Phys. Rev.* D **13** 2739
[6] Montonen C 1976 *Nucl. Phys.* B **112** 349
[7] Rajaraman R 1979 *Phys. Rev. Lett.* **42** 200
[8] Malik G P, Subba Rao J and Johri G 1985 *Pramana J. Phys.* **25** 123
[9] Whittaker E T and Watson G N 1952 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[10] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
[11] Magnus W and Oberhettinger F 1949 *Formulas and Theorems for the Special Functions of Mathematical Physics* (New York: Chelsea)